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Abstract. One of the primary motivations for using fixed field alternating gradient accelerators (FFAGs) is their ability to accelerate rapidly, since the magnetic fields do not need to be varied. However, one must then face the difficulty that the time of flight in an FFAG depends strongly on the particle energy. Traditionally, this is dealt with by varying the RF frequency. The rate at which one can vary the RF frequency is limited, and a cavity and power source which have a rapidly varying RF frequency are costly. One solution to this is harmonic number jump acceleration [Alessandro G. Ruggiero, Phys. Rev. ST Accel. Beams **9**, 100101 (2006)], where the RF frequency is fixed. The RF frequency is chosen so that each turn has an integer number of RF periods, but that integer number is different on each turn. When accelerating rapidly, a large number of cavities is often required. This paper will show that in general, the time of flight can only be an integer number of RF periods for all turns at one position in the ring. It will then compute how well one can do when cavities are distributed everywhere in the ring. The paper will show some examples, and will discuss possible applications for this technique.

Keywords: Fixed Field Alternating Gradient Accelerator; Harmonic Number Jump
PACS: 29.27.Bd

INTRODUCTION

One of the major problems that must be faced in an FFAG design is the variation of the time of flight with energy. In practice, there are two ways in which this problem is dealt with: the RF frequency is varied as the particles accelerate (generally the approach in scaling FFAGs), or the acceleration is done so rapidly that the particles do not have time to go far from the RF crest (generally the approach in highly relativistic non-scaling FFAGs).

If one varies the RF frequency, the rate at which one can accelerate will end up being limited by the rate at which one can vary the RF frequency. Furthermore, the low quality factor required for the cavity frequency to be varied at a high rate will result in a high peak power requirement for the cavity.

If one accelerates highly relativistic particles rapidly (10–20 turns, typical for muon FFAGs), one has three choices. The first is the approach taken in non-scaling FFAGs for muon acceleration: make the machine isochronous within the machine energy range, and use a high RF frequency [1, 2, 3, 4]. This appears to work well at high energies, but becomes less efficient at lower energies [5, 6], especially due to the variation of the time of flight with transverse amplitude [7]. The second approach is to use a scaling FFAG with a low RF frequency (one is forced to low frequencies due to the large time of flight variation with energy in scaling FFAGs [8, 9]). This presents problems with achieving the required high gradients in the low frequency cavities. A third approach is to use high frequency RF cavities in a scaling FFAG, and to use harmonic number jump acceleration, where the time of flight on each turn is a different number of RF periods [10, 11, 12].

This last approach, when used for accelerating muons, requires that the ring be almost completely filled with RF cavities to avoid excessive muon decays. This paper will show that doing so presents a difficulty with keeping all of the cavities synchronized with the particle beam. The paper will then show how to optimally choose the cavity frequencies and phases so as to minimize the negative impact of this lack of synchronization. Finally, the paper will discuss some ways to exploit this technique, and further work that needs to be done.

BASIC EQUATIONS

Assume that we have a number of cavities in the ring, enumerated by an integer k. The cavities are at position θ_k in the ring (distance along reference curve divided by circumference times 2π), with a maximum energy gain in the cavity V_k, RF frequency f_k, and phase ϕ_k. Furthermore, say that the the time of flight along a closed orbit with energy E is

$T(E)$. Then the time t and energy E of a particle at position θ in the ring are given by the equations of motion

$$\frac{dt}{d\theta} = \frac{T(E)}{2\pi} \qquad \frac{dE}{d\theta} = V(\theta)\cos\bigl(2\pi f(\theta)t + \phi(\theta)\bigr)$$
$$V(\theta) = \sum_k V_k \delta_{2\pi}(\theta - \theta_k) \qquad f(\theta_k) = f_k \qquad \phi(\theta_k) = \phi_k \qquad \theta_{2\pi}(\theta) = \sum_m \delta(\theta - 2\pi m) \tag{1}$$

The properties of the discrete cavities enumerated by k have been converted to functions of θ, since this will be useful for subsequent computations. Note that $V(\theta)$, $f(\theta)$, and $\phi(\theta)$ are all periodic functions of θ, whereas $t(\theta)$ and $E(\theta)$ are not in general periodic.

Single Cavity

Assume first that there is only a single cavity in the ring, which is located at $\theta_0 = \pi$. If one wishes to arrive at the cavity with the same phase each time, then one can write the energy at the opposite end of the ring from the cavity and the arrival time at the cavity as

$$E(2\pi k) = E(0) + kV_0\cos\phi_0 = E(0) + k\Delta E \qquad t(2\pi k + \pi) = \sum_k T\bigl(E(0) + k\Delta E\bigr) - \frac{1}{2}T\bigl(E(0)\bigr). \tag{2}$$

The equation for the energy uses the assumption that the particle arrives at the same RF phase on each pass. For the time equation to be consistent with this, one must have

$$fT\bigl(E(0) + k\Delta E\bigr) = h_k, \tag{3}$$

where h_k is an integer (the harmonic number).

This paper will only treat the case where $T(E)$ is a linear function of E:

$$T(E) = T\bigl(E(0)\bigr) + \Delta T[E - E(0)]/\Delta E. \tag{4}$$

One must deal with the fact that this is not the case, and that difficulty is discussed by other authors [12]. This paper is only trying to address the additional difficulty of the cavities being distributed around the ring, and therefore focuses on that difficulty by using a form for the time of flight that does not have additional difficulties associated with being a nonlinear function.

The simplification of the linearity of the time of flight with respect to energy results in Eq. (3) becoming

$$f[T\bigl(E(0)\bigr) + k\Delta T] = h_k. \tag{5}$$

Considering the difference between two of these equations with different k, one arrives at the condition

$$f\Delta T = m, \tag{6}$$

where m is an integer. Thus, for a single cavity, when the time of flight is a linear function of energy, harmonic number jump acceleration becomes straightforward.

Cavities Distributed Everywhere

Now, consider the case where cavities are distributed everywhere around the ring. I will take this to mean that the energy gain is uniformly distributed around the ring:

$$E(\theta) = E(0) + \Delta E\frac{\theta}{2\pi}. \tag{7}$$

Continuing to assume that the time of flight as a function of energy is given by Eq. (4), the time of flight at any given position in the ring is given by

$$t(\theta) = t(0) + T\bigl(E(0)\bigr)\frac{\theta}{2\pi} + \frac{\Delta T}{2}\left(\frac{\theta}{2\pi}\right)^2. \tag{8}$$

If one would like to arrive at the same RF phase on each turn,

$$f(\theta)[t(\theta+2\pi)-t(\theta)] = f(\theta)\left[T\big(E(0)\big)+\frac{\Delta T}{2}+\Delta T\frac{\theta}{2\pi}\right] = h(\theta), \tag{9}$$

where $h(\theta)$ is an integer.

One can use Eq. (9) to compute the cavity frequencies that are required to have the RF phase be identical on each turn:

$$f(\theta) = \frac{h(\theta)}{T\big(E(0)\big)+\frac{\Delta T}{2}+\Delta T\frac{\theta}{2\pi}}. \tag{10}$$

Furthermore, since the harmonic number $h(\theta)$ must be an integer at both θ and $\theta+2\pi$,

$$h(\theta+2\pi)-h(\theta) = f(\theta)\Delta T = m. \tag{11}$$

Combining Eqs. (10) and (11), one finds

$$\frac{m}{\Delta T} = \frac{h(\theta)}{T\big(E(0)\big)+\frac{\Delta T}{2}+\Delta T\frac{\theta}{2\pi}}. \tag{12}$$

The left hand side of this equation is a constant, whereas the right hand side depends on θ (it must, since $h(\theta)$ is an integer, and thus changes only in discrete steps, whereas the denominator of the right hand side is a continuous function of θ). This is a contradiction, and thus it is not possible to make the phases the same for every cavity in the ring.

APPROXIMATE PHASE MATCHING

Since one cannot make all of the RF phases the same, the next question is whether one can get sufficiently close to making the phases identical to operate the machine. This section will describe several methods for attempting this.

All Cavities with Identical Frequencies

The simplest thing one can do is to give all of the cavities the same frequency f. Meeting the harmonic number jump condition requires that $f\Delta T = m$. Assume that the cavity at $\theta = 0$ is synchronized, which means that

$$f[T\big(E(0)\big)+\Delta T/2] = h, \tag{13}$$

where h is an integer. For the cavity at $\theta = \pi$, the phase error divided by 2π is

$$f[T\big(E(0)\big)+\Delta T] = h+m/2. \tag{14}$$

For m an odd integer, the cavity at $\theta = \pi$ is completely out of phase (if m were even, one could have chosen a different point in the ring). This is undesirable since that turn would have no average acceleration, so the cavities must be given different frequencies.

Get First Turn Right

The next solution one can try is to choose the cavity frequencies so that the phases are identical on the first and second passes through the RF cavities. This will require setting the cavity frequencies to different values.

To make the RF phase the same for the first and second cavity passes, set the frequencies to

$$f(\theta) = \frac{h(0)}{t(\theta+2\pi)-t(\theta)} = \frac{h(0)}{T\big(E(0)\big)+\frac{\Delta T}{2}+\Delta T\frac{\theta}{2\pi}}, \tag{15}$$

where $\theta \in (0, 2\pi)$. One can then write down the number of RF periods at θ that occur between a reference particle passing through at θ and $\theta + 2\pi$,

$$h(\theta) = f(\theta)[t(\theta+2\pi) - t(\theta)] = h(0)\frac{T\big(E(0)\big) + \dfrac{\Delta T}{2} + \Delta T\dfrac{\theta}{2\pi}}{T\big(E(0)\big) + \dfrac{\Delta T}{2} + \Delta T\left(\dfrac{\theta}{2\pi} - \left\lfloor\dfrac{\theta}{2\pi}\right\rfloor\right)}. \tag{16}$$

Ideally, this should be an integer, but it clearly is not. By construction, it is equal to $h(0)$ for $\theta \in (0, 2\pi)$. Furthermore, if one chooses the frequency and revolution period appropriately, one can choose one angle where $h(\theta)$ is an integer on every turn. To do this, compute

$$h(\theta+2\pi) - h(\theta) = \frac{h(0)\Delta T}{T\big(E(0)\big) + \dfrac{\Delta T}{2} + \Delta T\left(\dfrac{\theta}{2\pi} - \left\lfloor\dfrac{\theta}{2\pi}\right\rfloor\right)}. \tag{17}$$

This is a periodic function of θ. Making this an integer m at $\theta = 2\pi n + \pi$,

$$m = \frac{h(0)\Delta T}{T\big(E(0)\big) + \Delta T}. \tag{18}$$

It is convenient to rewrite (17) in terms of m and $h(0)$:

$$h(\theta+2\pi) - h(\theta) = \frac{h(0)m}{h(0) + m\left(\dfrac{\theta}{2\pi} - \left\lfloor\dfrac{\theta}{2\pi}\right\rfloor - \dfrac{1}{2}\right)}. \tag{19}$$

The worst case error in the harmonic number difference (i.e., the difference of Eq. (19) from m) is

$$\frac{m^2}{2h(0) - m}. \tag{20}$$

This is the increase in the error in the harmonic number on each turn. The phase error is 2π times the sum of all of the errors in the harmonic number. Thus, after n turns, the worst phase error is

$$2\pi\frac{(n-1)(n-2)}{2}\frac{m^2}{2h(0) - m}. \tag{21}$$

If one were accelerating on-crest, this would mean that one has approximately $\sqrt{h(0)}/m$ turns before some cavities begin decelerating.

The resulting cavity phases are shown in Fig. 1. From this figure and Eq. (19), the phase error on a given cavity pass, averaged over all cavities, is approximately zero. Since in harmonic number jump acceleration, one is generally accelerating off-crest so as to have synchrotron oscillations [11, 12], the effect of the phase error is further reduced.

Note that the cavity frequencies are monotonically increasing (or decreasing, depending on the sign of ΔT) with θ. Thus this scenario works only for particles moving in one direction. This plan will therefore not work with both signs of particles counter-rotating in the ring, as envisioned in the acceleration scenarios for a neutrino factory or muon collider [13].

Further Improvements

One could follow the same procedure as above, but make the harmonic number correct in all cavities on the central turn instead of the first turn. After n turns, the worst case phase error would then be

$$\pi\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\frac{m^2}{2h(0) - m}. \tag{22}$$

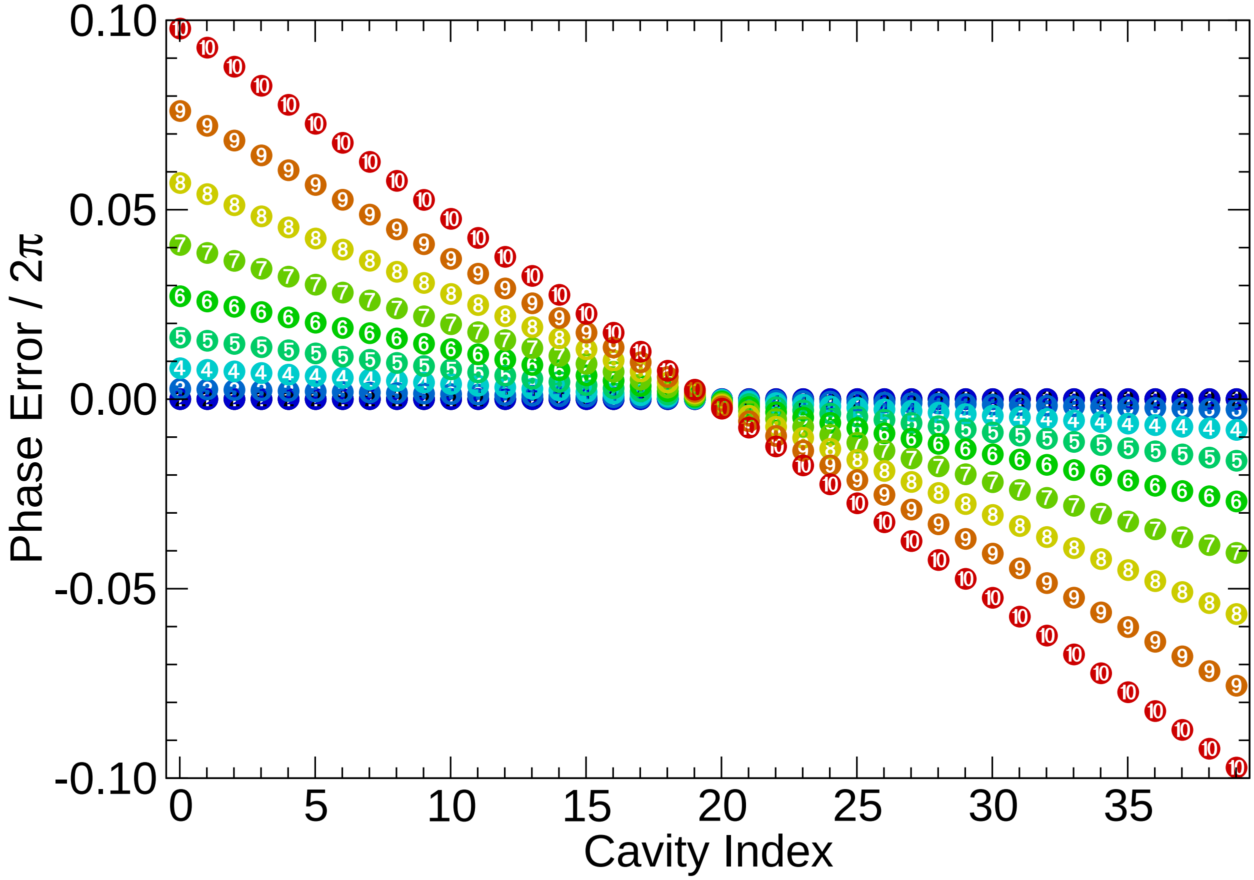

FIGURE 1. Phase error as a function of cavity index (position in the ring) and pass number (number in the circle, starting from 1). Cavity frequencies are set to get the first turn right and maintain the harmonic number between cavities 19 and 20 at an integer.

The resulting cavity phases are shown in Fig. 2. The maximum phase error is about 1/4 of what it is for the case where the frequencies are set correctly for the first turn. Or, equivalently, for a given phase error tolerance, one can accelerate for twice as many turns.

All of the previous scenarios have assumed that the phases on the first turn are set to their ideal values. If one averages over all the turns at a given cavity, the average phase error is greater than zero at some cavities and less than zero at others. One could instead adjust the initial cavity phases so that the average phase errors would be zero at each individual cavity. This is shown in Fig. 3. The maximum phase error is somewhat reduced from the case where the initial phases are set to zero, but not by a full factor of 2, due to increasing change in the phase error as one gets further from the turn where the frequencies are synchronized. Since the phase error at any cavity averaged over all turns is zero, and the phase error on a given turn averaged over all cavities is zero, one expects negligible effect from these phase errors as long as one is somewhat further away from the crest than the maximum phase error. If particles crossing the crest are an issue, it may be advantageous to minimize the maximum phase error at each cavity rather then making the average phase error zero at each cavity.

OTHER APPLICATIONS OF THE TECHNIQUE

At each individual cavity, one can select the turn at which the RF frequency is synchronized to the RF and the phase difference from the desired phase at some particlar turn. In the model where the time of flight depends linearly on the energy, one can even choose a single cavity which is synchronized with the RF at every turn. To minimize the maximum phase error, the previous examples have chosen the middle cavity to be synchronized. However, one could

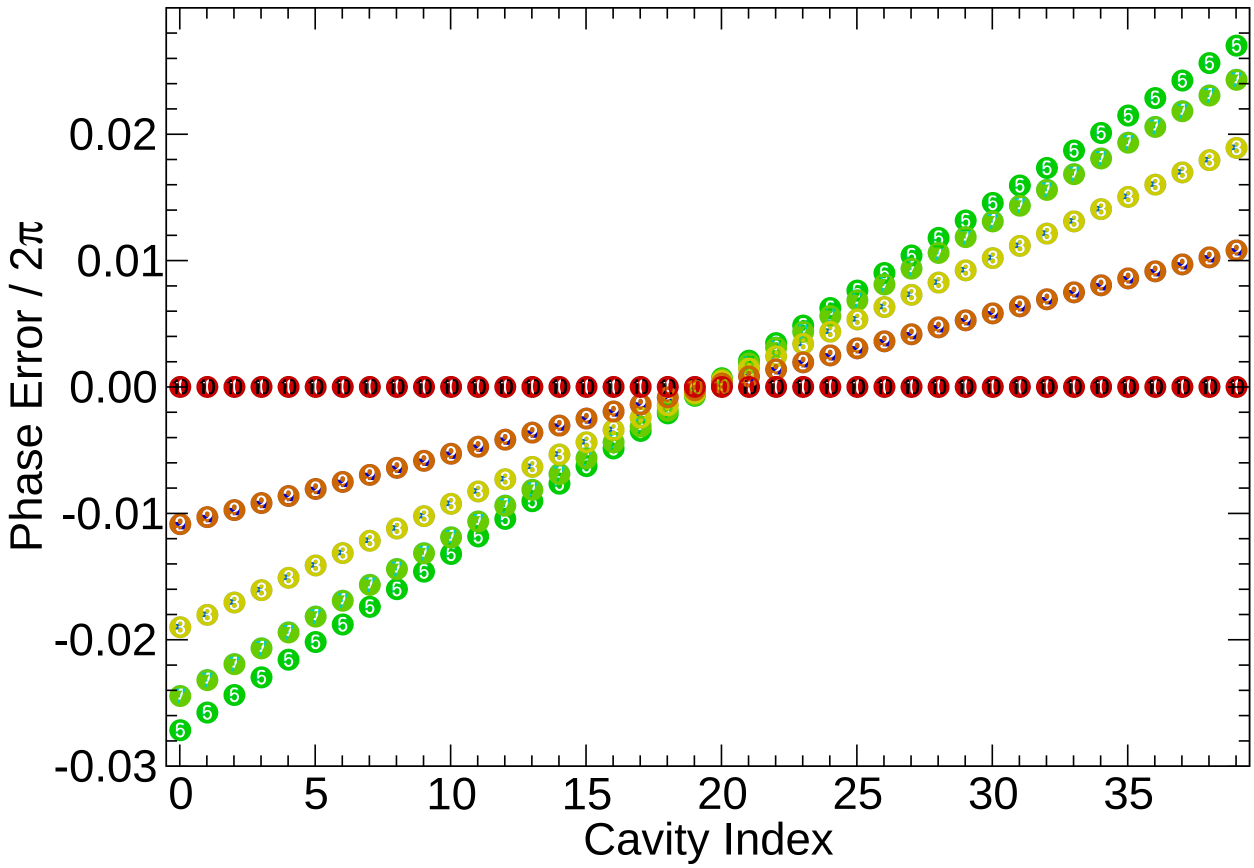

FIGURE 2. As in Fig. 1, except that the cavity frequencies are set to get the middle turn right instead of the first. Note that 10 cavity passes are shown here; the last 5 and the first 5 have identical phases.

synchronize a different cavity, as shown in Fig. 4. The RF phase on a given turn, averaged over all the cavities, is no longer zero. This means that the acceleration rate will vary with turn number.

There are three quantities that one can conceive of easily varying: which cavity is synchronized for all turns, which turn has zero phase error, and which turn has the right frequency for all cavities. These allow some degree of control over how the beam accelerates as a function of turn number.

One need not fill the ring uniformly with cavities. This will allow for a smaller phase error (if the cavities are all placed near each other) and therefore potentially more turns of acceleration. This, along with using the phases to vary the rate of acceleration with turn number, might be useful in the acceleration of protons, which require a relatively small number of cavities per turn, a larger number of turns, and an energy gain that varies with turn number [12].

FURTHER WORK AND CONCLUSIONS

The computations described here are far from complete, and are primarily meant to give a general idea of what needs to be done to make optimal use of harmonic number jump in a machine requiring large numbers of cavities to be installed. In particular, the following components are missing:

- There is no variation of energy gain with the arrival time.
- The time of flight should not be a linear function of the energy.
- The computation assumes only a single bunch; for a train of bunches, different bunches will see different phases.
- Particles other than the "reference" particle have not been considered. One must consider how the RF bucket is modified by the phase variation here. In particular, the large jump in phase from one "turn" to the next will likely

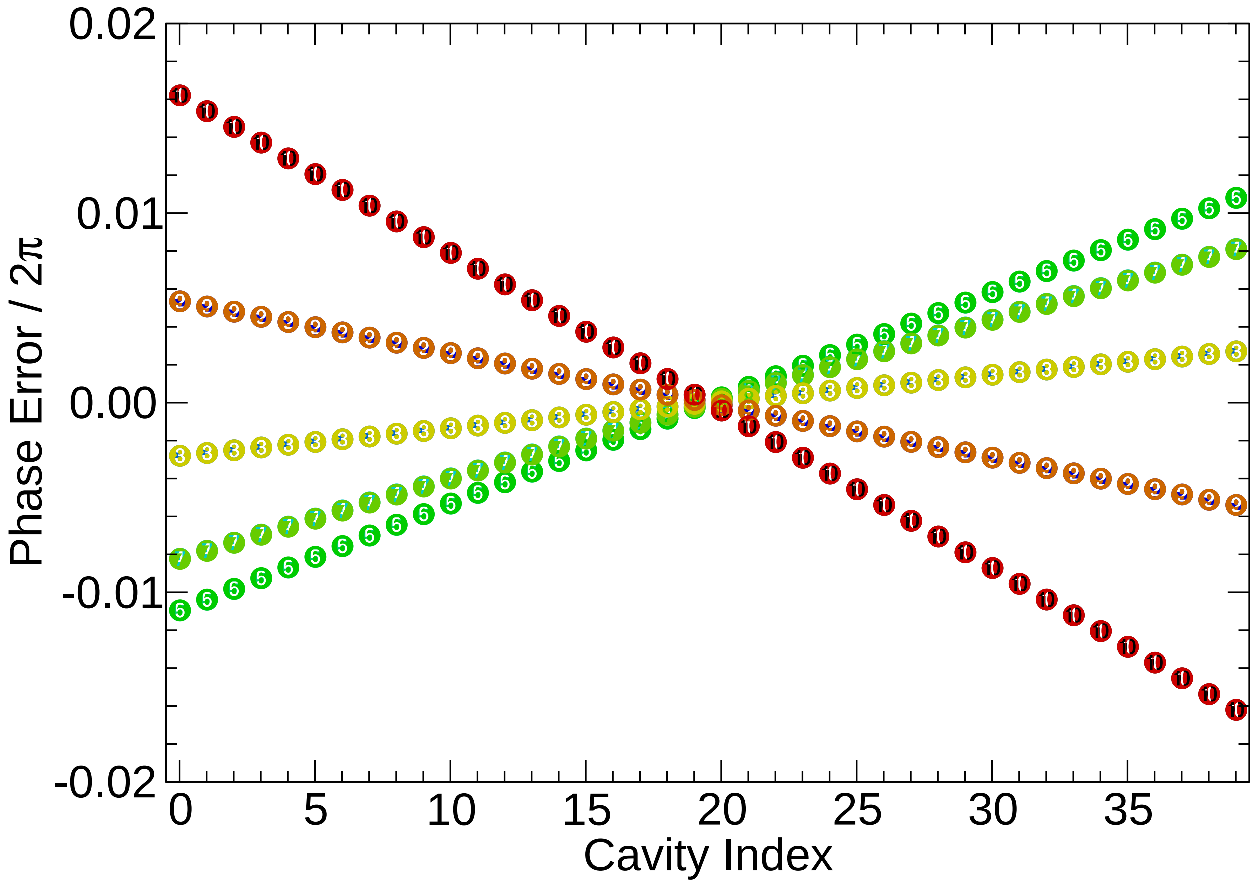

FIGURE 3. As in Fig. 2, except that the cavity phases are set to make the average phase error zero at each individual cavity.

have a significant effect on the longitudinal dynamics.

Nonetheless, this paper has demonstrated some important results. First, that when the ring is filled with cavities, the RF phase must vary from one turn to the next, even when the time of flight depends linearly on energy. However, by choosing the cavity phases and frequencies correctly, one can make the effect of this variation relatively small, at least for a limited number of turns. One may even be able to take advantage of the phase variation to vary the energy gain with turn number to account for the fact that the time of flight does not depend linearly on energy. Clearly, to analyze the complete system, there is significantly more work to be done.

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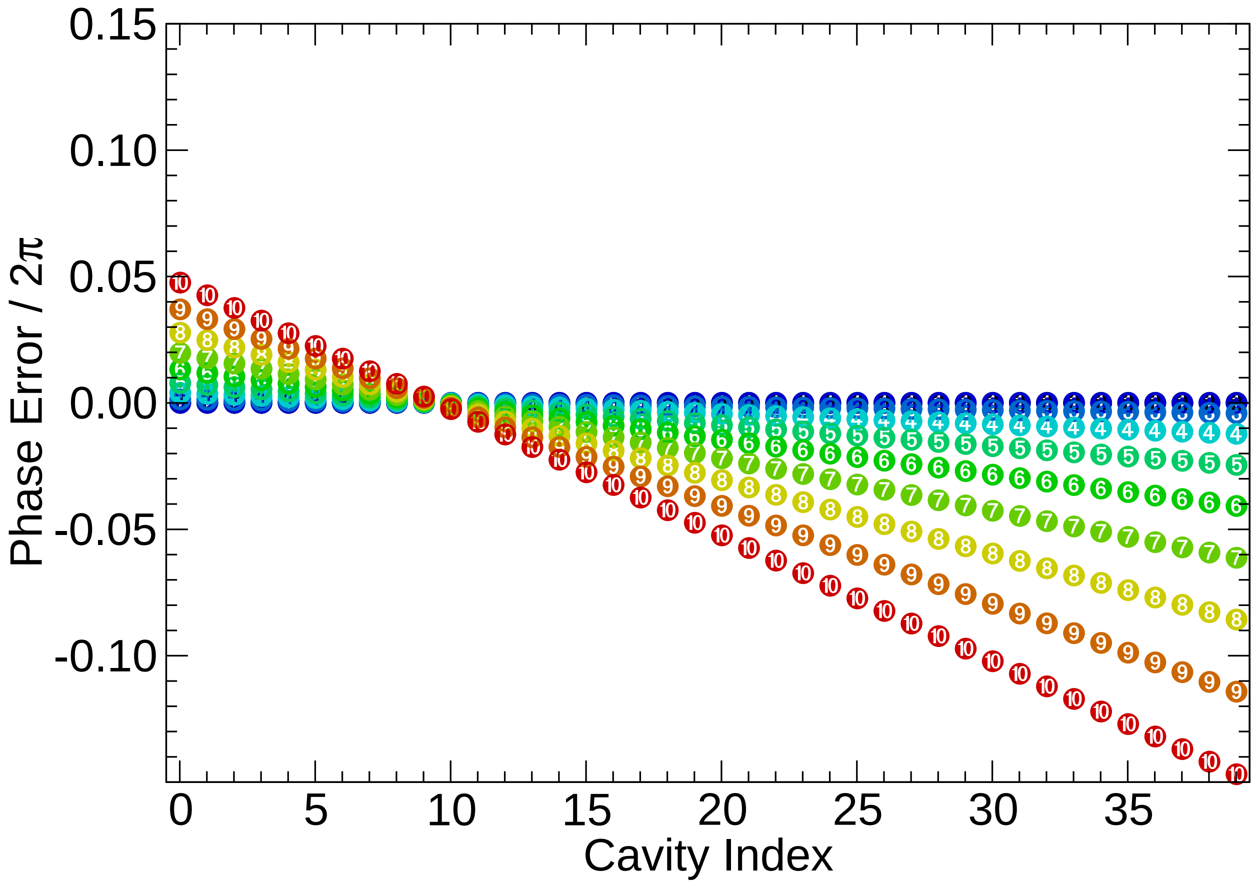

FIGURE 4. As in Fig. 1, except that the point between cavities 9 and 10 is synchronized on every turn.

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